

SMALL SAMPLE TESTS

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STUDENT'S t DISTRIBUTION

- When the original sampled population is normal, $\bar{x}$ and $z = (\bar{x} - \mu) / (\sigma / \sqrt{n})$ both have normal distributions, for any sample size.
- When the original sampled population is not normal, $\bar{x}$, $z = (\bar{x} - \mu) / (\sigma / \sqrt{n})$, and $z \approx (\bar{x} - \mu) / (s / \sqrt{n})$ all have approximately normal distributions, if the sample size is large.
- But when the sample size n is small, the statistic $(\bar{x} - \mu) / (s / \sqrt{n})$ does not have a normal distribution.
- An Englishman named W.S. Gosset in 1908, derived a formula for the density function of this statistic.

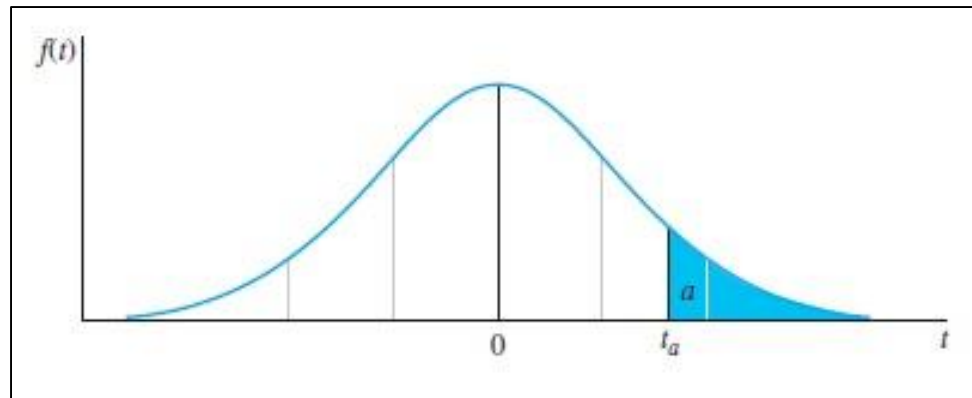
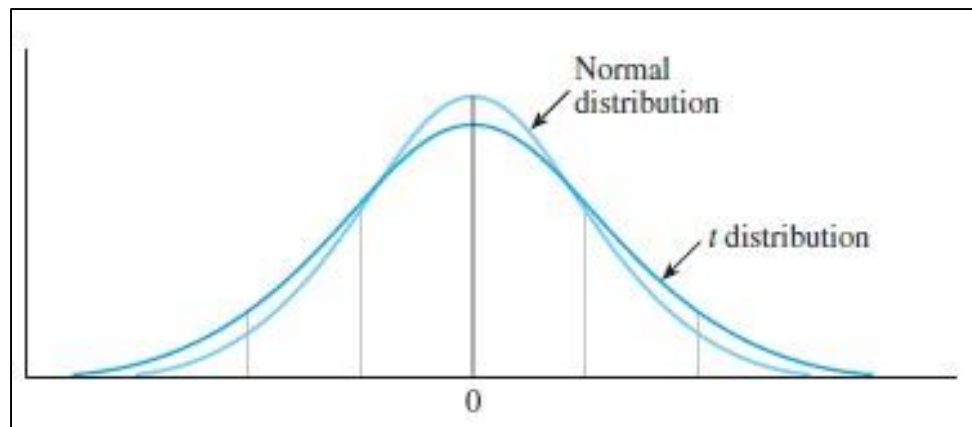
STUDENT'S t DISTRIBUTION

- The statistic $t = (\bar{x} - \mu) / (s / \sqrt{n})$ for random samples of size n from a normal population, follows Student's t distribution with $(n-1)$ degrees of freedom.
- Characteristics:
 - ▣ It is mound-shaped and symmetric about $t = 0$.
 - ▣ It is more variable than z , with “heavier tails”.
 - ▣ The shape of the t distribution depends on the sample size n . When n is infinitely large, the t and z distributions are identical.

STUDENT'S t DISTRIBUTION

- Assumptions:
 - ▣ The sample must be randomly selected.
 - ▣ The population from which the sample is drawn must be normally distributed.
- The divisor $(n - 1)$ in the formula for the sample variance s^2 is called the number of degrees of freedom (df) associated with s^2 . It determines the shape of the t distribution. It refers to the number of independent squared deviations in s^2 that are available for estimating σ^2 .

STUDENT'S t DISTRIBUTION



SMALL-SAMPLE HYPOTHESIS TEST FOR μ

- Null hypothesis: $H_0 : \mu = \mu_0$
- Alternative hypothesis:
 - ▣ One-Tailed Test $H_1 : \mu > \mu_0$ or $H_1 : \mu < \mu_0$
 - ▣ Two-Tailed Test $H_1 : \mu \neq \mu_0$
- Test statistic: $t = (\bar{x} - \mu_0) / (s / \sqrt{n})$
- Rejection region: Reject H_0 when
 - ▣ $t > t_\alpha$ or $t < -t_\alpha$ (for one-tailed test)
 - ▣ $t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$ (for two-tailed test)
- The critical values of t , t_α and $t_{\alpha/2}$ are based on $(n-1)$ degrees of freedom.

SMALL-SAMPLE $(1-\alpha)100\%$ CONFIDENCE INTERVAL FOR μ

□
$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $s/\sqrt{n}$ is the standard error of the mean.

TEST FOR THE DIFFERENCE BETWEEN TWO MEANS: INDEPENDENT RANDOM SAMPLES

- Null hypothesis: $H_0 : (\mu_1 - \mu_2) = D_0$, where D_0 is some specified difference that you wish to test. For no difference between μ_1 and μ_2 ; $D_0 = 0$.
- Alternative hypothesis:
 - ▣ One-Tailed Test $H_1 : (\mu_1 - \mu_2) > D_0$ or $(\mu_1 - \mu_2) < D_0$
 - ▣ Two-Tailed Test $H_1 : (\mu_1 - \mu_2) \neq D_0$
- Test statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{s^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \text{ where } s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

TEST FOR THE DIFFERENCE BETWEEN TWO MEANS: INDEPENDENT RANDOM SAMPLES

- Rejection region: Reject H_0 when
 - ▣ $t > t_\alpha$ (or $t < -t_\alpha$) (for one-tailed test)
 - ▣ $t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$ (for two-tailed test)
- The critical values of t , t_α , and $t_{\alpha/2}$ are based on $(n_1 + n_2 - 2)$ df. The tabulated values can be found using Table of the **Student's t distribution**.
- Assumptions:
 - ▣ The samples are randomly and independently selected from normally distributed populations.
 - ▣ The variances of the populations σ_1^2 and σ_2^2 are equal.

$(1-\alpha)100\%$ CONFIDENCE INTERVAL FOR $(\mu_1-\mu_2)$ BASED ON INDEPENDENT RANDOM SAMPLES

$$\left(\overline{x}_1 - \overline{x}_2\right) \pm t_{\alpha/2} \sqrt{s^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}$$

where s^2 is the pooled estimate of σ^2 .

PAIRED-DIFFERENCE TEST OF HYPOTHESIS

- Sometimes the observations are taken in pairs from two samples so that the independence of two samples assumption doesn't hold.
- This pairing is based on some common characteristics between the variables.
- Such experimental design is called a paired-difference or matched pairs design.
- The paired-difference test or matched pairs design is an example of an experimental design called a randomized block design.

PAIRED-DIFFERENCE TEST OF HYPOTHESIS

- When the experimental units are quite heterogeneous then, groups of relatively similar experimental units called **blocks** are formed before starting the experiment.
- This ensures that the true difference between the procedures is not masked by the heterogeneity among the experimental units.
- Pairing or blocking is decided at the stage of planning of the experiment, and not after the data are collected.
- Examples:
 - ▣ An experimenter may choose to use pairs of identical twins to compare two learning methods.
 - ▣ A physician may record a patient's blood sugar level before and after a new diabetic diet is given.

PAIRED-DIFFERENCE TEST OF HYPOTHESIS FOR $(\mu_1 - \mu_2) = \mu_d$: DEPENDENT SAMPLES

- Null hypothesis: $H_0 : \mu_d = 0$
- Alternative hypothesis:
 - ▣ One-Tailed Test $H_1 : \mu_d > 0$ or $H_1 : \mu_d < 0$
 - ▣ Two-Tailed Test $H_1 : \mu_d \neq 0$
- Test statistic:

$$t = \frac{\bar{d}}{s_d / \sqrt{n}}$$

where n = Number of paired differences, $\bar{d}$ = Mean of the sample differences, s_d = Standard deviation of the sample differences.

PAIRED-DIFFERENCE TEST OF HYPOTHESIS FOR $(\mu_1 - \mu_2) = \mu_d$: DEPENDENT SAMPLES

- Rejection region: Reject H_0 when
 - ▣ $t > t_\alpha$ (or $t < -t_\alpha$) (for one-tailed test)
 - ▣ $t > t_{\alpha/2}$ or $t < -t_{\alpha/2}$ (for two-tailed test)
- The critical values of t , t_α and $t_{\alpha/2}$ are based on $(n-1)$ *df*. The tabulated values can be found using Table of the **Student's t distribution**.

$(1-\alpha)100\%$ CONFIDENCE INTERVAL FOR $(\mu_1-\mu_2) = \mu_d$ BASED ON A PAIRED-DIFFERENCE EXPERIMENT

$$\bar{d} \pm t_{\alpha/2} \left(\frac{S_d}{\sqrt{n}} \right)$$

- Assumptions: The experiment is to be designed as a paired-difference test so that the n differences represent a random sample from a normal population.

Chi-square

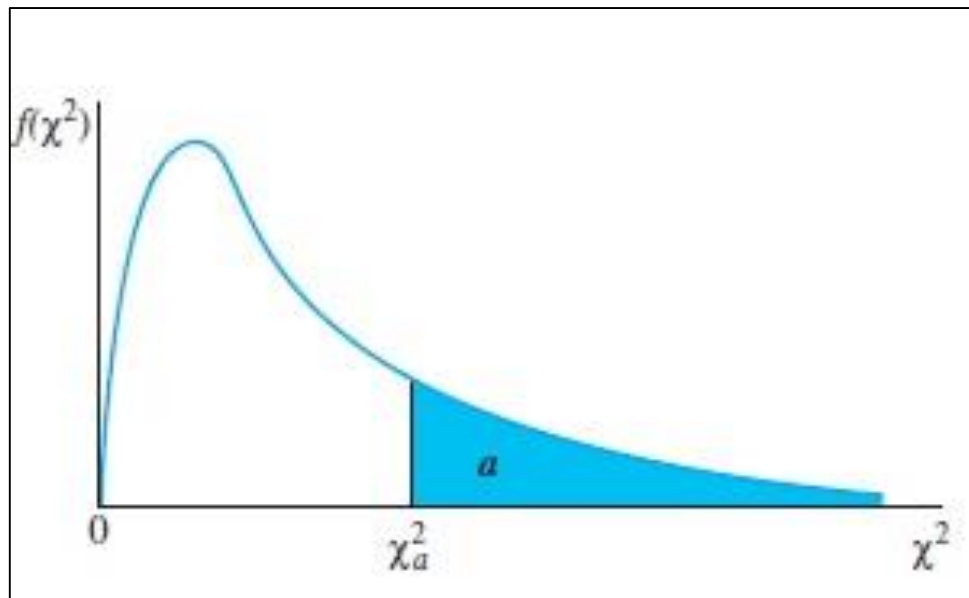
- The standardized statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

is called a chi-square variable and has a sampling distribution called the chi-square probability distribution, with $(n-1)$ degrees of freedom.

Chi-square

- The distribution is asymmetric and starts at 0.
- The shape of the distribution varies with the sample size n or the degrees of freedom $(n-1)$ associated with s^2 .



TEST OF HYPOTHESIS FOR POPULATION VARIANCE σ^2

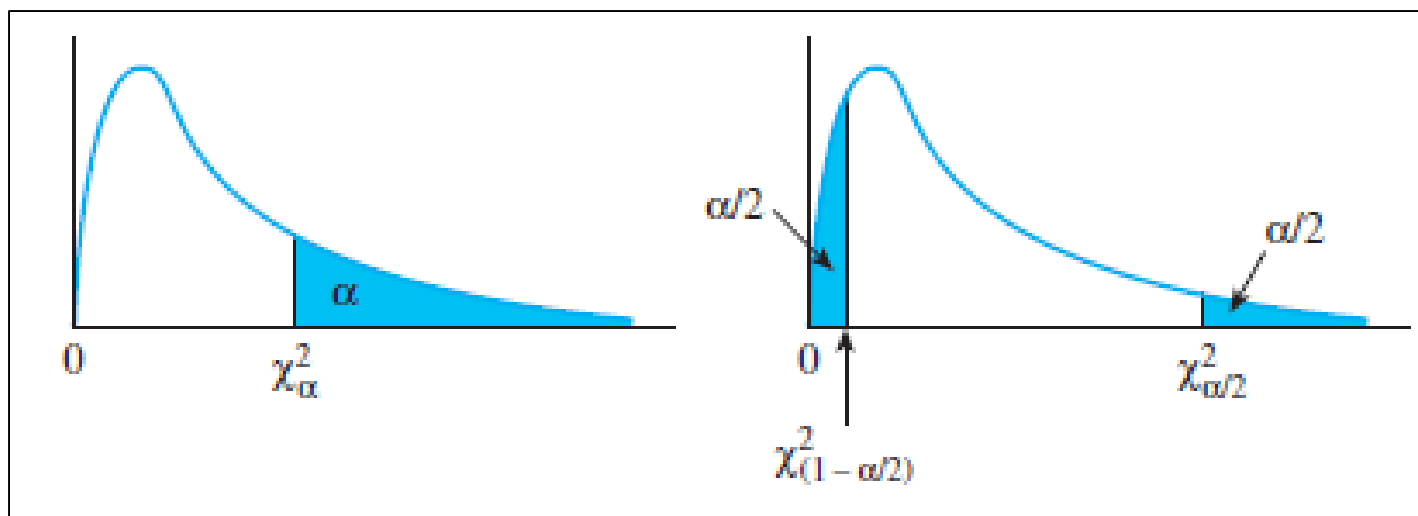
- Null hypothesis: $H_0 : \sigma^2 = \sigma_0^2$
- Alternative hypothesis:
 - ▣ One-Tailed Test $H_1 : \sigma^2 > \sigma_0^2$ ($\sigma^2 < \sigma_0^2$)
 - ▣ Two-Tailed Test $H_1 : \sigma^2 \neq \sigma_0^2$
- Test statistic:

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

TEST OF HYPOTHESIS FOR POPULATION VARIANCE σ^2

- Rejection region: Reject H_0 when
 - $\chi^2 > \chi^2_{\alpha}$ (or $\chi^2 < \chi^2_{1-\alpha}$) (for one-sided test)
 - $\chi^2 > \chi^2_{\alpha/2}$ or $\chi^2 > \chi^2_{(1-\alpha/2)}$ (for two-sided test)
- The critical values of χ^2 are based on $(n-1)$ df. These tabulated values can be found using **Chi-Square Table**.

TEST OF HYPOTHESIS FOR POPULATION VARIANCE σ^2



$(1 - \alpha)100\%$ CONFIDENCE INTERVAL FOR σ^2

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{(1-\alpha/2)}}$$

Where $\chi^2_{\alpha/2}$ and $\chi^2_{(1-\alpha/2)}$ are the upper and lower χ^2 values, which locate one-half of α in each tail of the chi-square distribution.

- Assumption: The sample is randomly selected from a normal population.

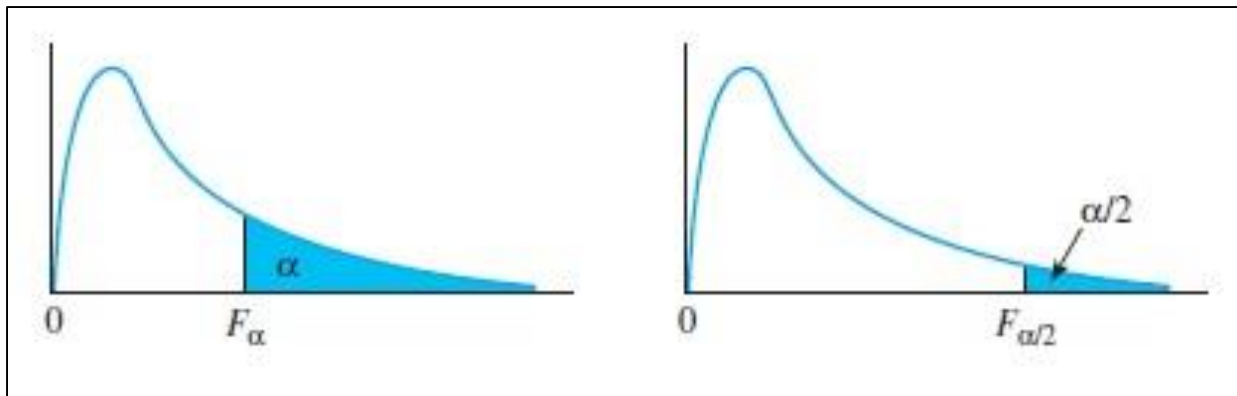
F Distribution

- When independent random samples are drawn from two *normal populations with equal variances*—that is, $\sigma_1^2 = \sigma_2^2$ then the statistic (s_1^2/s_2^2) has a probability distribution in repeated sampling that is known as an ***F distribution***.
- The shape of the *F distribution* is *asymmetric* and depends on the number of degrees of freedom associated with s_1^2 and s_2^2 , represented as $df_1 = (n_1 - 1)$ and $df_2 = (n_2 - 1)$, respectively.

F Distribution

□ Assumptions:

- Random and independent samples are drawn from each of two normal populations.
- The variance in the two populations is the same and can be measured by a common variance, σ^2 ; i.e., $\sigma_1^2 = \sigma_2^2 = \sigma^2$.



TEST OF HYPOTHESIS FOR EQUALITY OF TWO POPULATION VARIANCES

- Null hypothesis: $H_0 : \sigma_1^2 = \sigma_2^2$
- 2. Alternative hypothesis:
 - ▣ One-Tailed Test $H_1 : \sigma_1^2 > \sigma_2^2$ or $\sigma_1^2 < \sigma_2^2$
 - ▣ Two-Tailed Test $H_1 : \sigma_1^2 \neq \sigma_2^2$
- Test statistic

$$F = \frac{S_1^2}{S_2^2}$$

- Rejection region: Reject H_0 when
 - ▣ $F > F_\alpha$ (for one-tailed test)
 - ▣ $F > F_{\alpha/2}$ (for two-tailed test)

TEST OF HYPOTHESIS FOR EQUALITY OF TWO POPULATION VARIANCES

- The critical values of F_α and $F_{\alpha/2}$ are based on $df_1 = (n_1 - 1)$ and $df_2 = (n_2 - 1)$. These tabulated values can be found using the **F Distribution tables**.
- Assumptions: The samples are randomly and independently selected from normally distributed populations.

CONFIDENCE INTERVAL FOR σ_1^2 / σ_2^2

$$\square \frac{s_1^2}{s_2^2} \frac{1}{F_{df_1, df_2}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{s_1^2}{s_2^2} F_{df_2, df_1}$$

where $df_1 = (n_1 - 1)$ and $df_2 = (n_2 - 1)$. F_{df_1, df_2} is the tabulated critical value of F corresponding to df_1 and df_2 degrees of freedom in the numerator and denominator of F respectively, with area $\alpha/2$ to its right.

- Assumptions: The samples are randomly and independently selected from normally distributed populations.

Failing to meet assumptions.....

- If the data doesn't satisfy the small sample assumptions then:
 - ▣ Either select relatively large samples and use one of the large sample procedures which do not rely on the normality or equal variance assumptions.
 - ▣ Or use a nonparametric test because these tests are specifically developed so that few or no distributional assumptions are required for their use.



Thanks